Engineering Notes

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Attitude Dynamics of an Orbiting Electromagnet

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Introduction

THE nature and interaction of high-energy particles cannot be studied on the Earth's surface because of limitations on size and power of existing particle accelerators. The concept of studying the behavior of high-energy cosmic particles of 10^{14}–10^{15} ev or higher by means of orbiting vehicles is an attractive one, since these vehicles carry the equipment for capture, detection, and observation. In one of the experimental setups considered[1] the vehicle will carry a huge electromagnet.

This Note deals with the attitude dynamics of such a vehicle. Analysis indicates that enormous torques are created by field interaction, as a result of which maintaining the vehicle in any fixed orientation in space is a formidable task. The vehicle's dipole oscillates conically about the Earth's magnetic field at an amplitude roughly equal to the initial misalignment. Our analytical investigation uses a simplified model of the Earth's magnetic field; this is followed by a more realistic approach, using a digital subroutine of the Earth's field furnished by NASA. The complete equations, including nonlinear terms, have been solved on an IBM 360/75 computer.

A Simplified Model of Attitude Dynamics

The model was inspired by Ref. 2 (pp. 10–14). The declination of the field was disregarded and the assumption made that the orbit (inclined at λ_o with respect to the equatorial plane) is circular. The field encountered varies as the second harmonic of the polar angle measured from the point of maximum latitude $\lambda = \lambda_o$. The triad $x_o y_o z_o$ in Fig. 1 follows the vehicle with y_o coinciding with the local vertical and z_o normal to the orbital plane. The magnetic field is defined approximately by

$$\begin{aligned} H_{x_o} &\approx 0 \\ H_{y_o} &\approx -0.7(1-\sin\lambda_o \cos^2\omega_o t) \\ H_{z_o} &\approx 0.4(\sin^2\omega_o t+\cos\lambda_o\cos^2\omega_o t) \end{aligned} \tag{1}$$

where ω_o is the orbital angular velocity of the vehicle. H is expressed in gauss. The vehicle weighs 150,000 lb (6.8×10^4 kg). The radius of gyration along its dipole moment is 7 ft (2.1 m); perpendicular to the axis it is of the order of 1.7 m. The principal moments of inertia are accordingly

$$I_1 = I_2 = 1.92\times10^5 \text{ kg-m}^2$$
and
$$I_3 = 3.1\times10^5 \text{ kg-m}^2 \tag{2}$$

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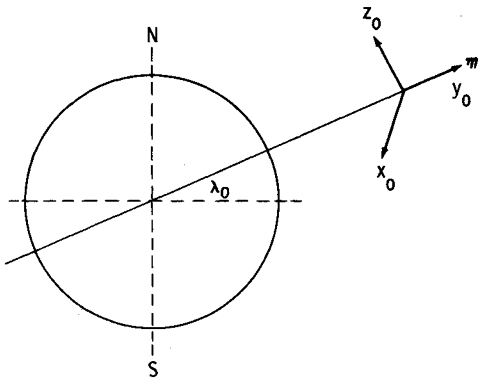

Fig. 1 Desired orientation with dipole (π) pointing radially outward from Earth.

along the dipole axis of the magnet. Initially the orientation of the dipole axis coincides with the local vertical y_o (see Figs. 1 and 2), while a new triad (ξ, η, ζ) is oriented as a result of a set of small angular deflections α, β, γ about the $x_o y_o z_o$ axes. We have

$$\begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \times \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{3}$$

The dipole moment referred to xyz is given by

$$\begin{aligned} \mathscr{M}_x &= -\gamma\mathscr{M} \\ \mathscr{M}_y &= \mathscr{M} \\ \mathscr{M}_z &= \alpha\mathscr{M} \end{aligned} \tag{4}$$

where the subscript zero has been dropped.

The torque $\mathbf{\Gamma}$ exerted upon the vehicle by the magnetic field $\mathbf{H}$ of the Earth is given by the cross product $\mathscr{M}\times\mathbf{H}$ or

$$\begin{aligned} \Gamma_x &= \mathscr{M}(H_z-\alpha H_y) \\ \Gamma_y &= \mathscr{M}(\alpha H_x+\gamma H_z) \\ \Gamma_z &= \mathscr{M}(-\gamma H_y-H_x) \end{aligned} \tag{5}$$

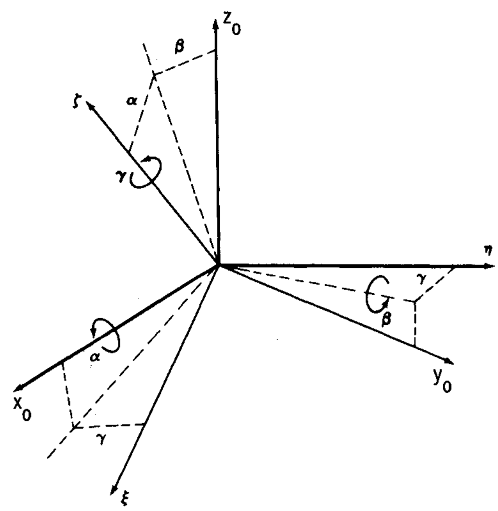

Fig. 2 Relations between the instantaneous body fixed axes (ξ, η, ζ) and the initial orientation (x_o, y_o, z_o).

The equations of attitude motion (Ref. 4, p. 140), with quadratic terms and H_x neglected, become

$$I_x \ddot{\alpha} + \mathscr{M} H_y \alpha + I_x \omega_o \dot{\beta} = \mathscr{M} H_z$$
$$I_y \ddot{\beta} - I_x \omega_o \dot{\alpha} - \mathscr{M} H_z \gamma = 0 \tag{6}$$
$$I_z \ddot{\gamma} + \mathscr{M} H_y \gamma = 0$$

The angular velocity of response ω_1 to the magnetic field is much faster than ω_o. Indeed for a volume V occupied by the magnet, we have $\mathscr{M} = BV/4\pi$. Taking $B = 66$ kilogauss, a 2-m-diam, and a thickness of 20 cm, we see that

$$\mathscr{M} = (6.6 \times 10^4 \times 10^4 \pi \times 20/4\pi) = 3.3 \times 10^9 \text{ cgs units}$$

and as a result

$$\omega_1 = (\mathscr{M}H/I)^{1/2} = 3.2 \times 10^{-2} \text{ rad/sec}$$

as compared to $\omega_o = 2\pi/5400 = 1.16 \times 10^{-3}$ rad/sec.

Equations (6) reveal that as expected β occurs only by its first and second time-derivatives. On the other hand, γ oscillates around zero with no forcing function. The second equation yields approximately

$$I_x \ddot{\alpha} + \mathscr{M} H_y \alpha = \mathscr{M} H_z + \text{const of integration}$$

Then α becomes determined by the ratio of H_z/H_y as anticipated. We see that if an active control system is desired, the torque of compensation may amount to the enormous value of 23 kg-m/rad for small angular misalignments.

The magnetic field of the Earth varies as a function of time in synchronism with the orbital motion of the vehicle; furthermore, the natural frequency of oscillation of the vehicle is about an order of magnitude higher than the orbital frequency. Disregarding the resulting simplification followed previously, a more accurate statement of the problem is possible. It is expressed by Mathieu-type equations not investigated here, as neither the time constant nor the steady-state behavior of the vehicle are substantially influenced by the higher accuracy, especially since the magnetic field of the Earth is not known with a comparable precision.

The contribution by gravity gradients is small and has also been neglected. Indeed, at the surface of the Earth, its value is of the order of 3.2×10^{-6} sec^{-2} and yields a torque applied to the vehicle equal to 7.6×10^{-2} kg-m, a value by far below the magnetic torques experienced.

Damping is applied to γ by known conventional means. It is reasonable to assume therefore that the steady-state value of γ is nil.

Refinement of our Dynamic Model

A more exact model was investigated by computer simulation. We assumed a 150,000 lb homogeneous cylinder as shown in Fig. 3 with a magnetic dipole strength of $\mathscr{M} = 3.3 \times 10^9$ ab-amp-cm^2 pointed along the η axis.

The equations of motion of the vehicle are given by

$$\mathbf{\Gamma} = (d\mathbf{L}/dt) + \boldsymbol{\omega} \times \mathbf{L}$$

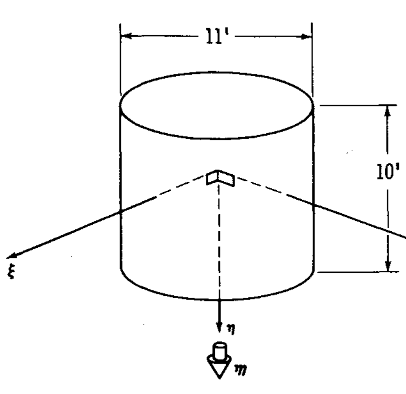

Fig. 3 Axes fixed in the body.

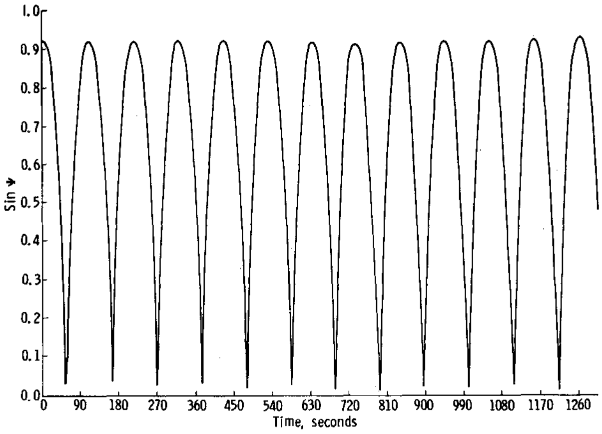

Fig. 4 Sine of angle between the Earth's magnetic field (H) and vehicle's dipole (π) vs time. Initial misalignment: $\sin\psi = 0.927$.

with an angular momentum expressed by the equation

$$\mathbf{L} = I(\boldsymbol{\omega} + \dot{\boldsymbol{\theta}})$$

The external torque amounts to $\mathbf{\Gamma} = \mathscr{M} \times \mathbf{H}$ where the Earth's field $\mathbf{H}$ was taken from Ref. 3. The equations of motion were solved with the help of an IBM 360/75 digital computer. The orbital parameters used in the simulation were: 25° orbital inclination, zero eccentricity, zero ascending node, zero argument of perigee, and a 250 naut mile altitude above the Earth. All cases were run with zero initial angular momentum.

The results presented in Figs. 4 and 5 are plots of $\sin\psi$ vs time for various initial values of ψ_o where

$$\psi = \sin^{-1}[(H_\xi{}^2 + H_\zeta{}^2)^{1/2}/|\mathbf{H}|]$$

is the angle between the vehicle's dipole and the Earth's magnetic field.

One can say that the vehicle's dipole will point substantially in the direction of the Earth's local magnetic field vector[3] and will oscillate about that field vector as shown in Figs. 4 and 5.

Conclusions

1) The magnet will oscillate about the Earth's local magnetic field with an amplitude determined by the initial misalignment. 2) Even in case of a perfect alignment, an oscillation of a few degrees with respect to H will take place. 3) A requirement of inertial stabilization of pointing would require the application of a torque far in excess of any practically realizable magnitude and calls for a redesign of the system with some built-in compensation.

References

[1] Alvarez, L. W., Anderson, J. A., and Buffington, A., "Cosmic Ray Studies with a Superconducting Magnet in a Space Station Facility," Space Sciences Laboratory Rept. XBL 696, July 8, 1969, University of California, Berkeley, Calif.

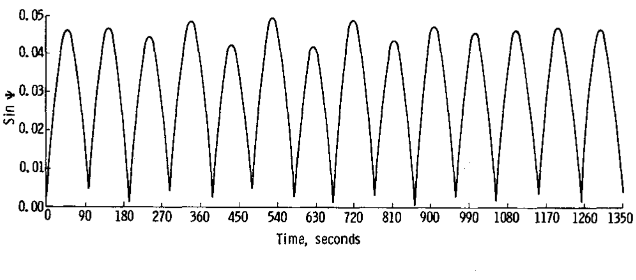

Fig. 5 Sine of angle between the Earth's magnetic field (H) and vehicle's dipole (π) vs time. Initial misalignment: $\sin\psi = 0.00267$.

[2] *Handbook of Geophysics*, Air Force Cambridge Research Center, Cambridge, Mass., 1957.

[3] Shapiro, M. and Bigliani, R., "Introduction to the Kinematic Attitude Orientation (KAO) Digital Computer Program System," ARP 250-009, Jan. 15, 1969, Grumman Aerospace Corp., Bethpage, N.Y.

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Analytical Solution for Extensible Tethers

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THIS Note presents an analytical approximation to the solution for an *extensible tether* problem. The mathematical model considers a large particle (m_1), such as a space station, and a smaller particle (m_2), connected by an ideal, massless tether—one incapable of sustaining other than tensile loads.

For this solution, the main particle (m_1) is assumed to move on a circular orbit (r_1), at an angular rate $\dot{\phi}_1$. Both particles are constrained to a single plane of motion, with the smaller mass acted upon by the central mass (μ) gravitational attraction and the tether tension. The geometric description for this case is shown in Fig. 1.

Equations of Motion

Under the assumptions made of this problem: $F \equiv |\bar{F}_1| = |\bar{F}_2|$, $\dot{\phi}_1$ is the angular motion for the tethered mass system; the line of action for the tether tension is parallel to $\bar{l}$, and $\pm\theta$ locates the tether ($\bar{l}$) with respect to the *local* radial direction ($\bar{e}_x$). The local triad ($\bar{e}_x, \bar{e}_y, \bar{e}_z$) moves with the main particle, m_1. The mass m_1 moves at speed $V_1 (\equiv r_1 \dot{\phi}_1)$ in a direction parallel to $\bar{e}_y$, at each instant.

Based on the abovementioned assumptions and descriptions, the vector equation of motion for m_2, with respect to m_1, can be written as

$$\ddot{\bar{l}} = \dot{\phi}_1{}^2[(1-\Delta^{-3})\bar{r}_1 - \Delta^{-3}\bar{l}] - \frac{F\bar{l}}{\tilde{m}l} \qquad (1)$$

Herein, because of the circular orbit, $\dot{\phi}_1{}^2 = \mu/r_1{}^3$,

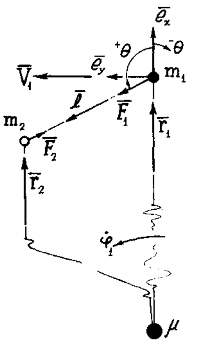

Fig. 1 Problem geometry. Unit vectors $\bar{e}_x$, $\bar{e}_y$ define the plane of motion.

$\tilde{m} \equiv m_1 m_2/(m_1+m_2)$ is the reduced mass parameter. Also, the quantity

$$\Delta \equiv [1+2(l/r_1)\cos\theta+(l^2/r_1{}^2)]^{1/2} \qquad (2)$$

is a parameter indicative of gravity gradient for this situation.

Next, introducing the dimensionless quantities: $\lambda \equiv l/r_1$, $\tau \equiv F/\tilde{m}/r_1\dot{\phi}_1{}^2$, and, changing the independent variable from t to ϕ (i.e., $\phi \equiv \dot{\phi}_1 t$), then the scalar differential equations defining the problem are, from Eq. (1)

$$\lambda'' = \lambda(1+\theta')^2+(1-\Delta^{-3})\cos\theta-\lambda\Delta^{-3}-\tau \qquad (3a)$$

and

$$\lambda\theta'' = -2\lambda'(1+\theta')-(1-\Delta^{-3})\sin\theta$$

In these, the primes denote differentiation with respect to ϕ.

For this study the angle θ is set at a fixed value, for any particular solution; consequently Eqs. (3) reduce to the set

$$\lambda'' = (\lambda+\cos\theta)(1-\Delta^{-3})-\tau \qquad (4a)$$

and

$$\lambda' = -(1-\Delta^{-3})(\sin\theta/2) \qquad (4b)$$

As an aid to obtaining the analytical solution desired here, since λ is a small term, we introduce the approximation

$$\Delta^{-3} \cong 1-3\lambda\cos\theta+\text{HOT}$$

correspondingly, Eqs. (4) are further reduced to

$$\lambda' \cong -\tfrac{3}{4}\lambda\sin 2\theta \qquad (5a)$$

and

$$\tau \cong (3\lambda/2)(1+\cos 2\theta)-\lambda'' \qquad (5b)$$

A more convenient "tether-length variable" can be had by introducing the dimensionless maximum value of $\lambda(\equiv\lambda_m)$. This leads to a new parameter replacing λ, namely

$$\sigma \equiv \lambda/\lambda_m = l/l_{max} \qquad (6a)$$

It is recognized that this new variable is constrained to

$$0 \leq \sigma \leq 1.0$$

during any tether *extension*, or *contraction*.

Next, for compatibility, define a new tension parameter, i.e., let

$$\tau_\sigma \equiv \frac{\tau}{\lambda_m} = \frac{F/\tilde{m}}{\dot{\phi}_1{}^2 l_{max}} \qquad (6b)$$

and rewrite Eqs. (5) accordingly.

The Solution

Remembering that θ is a fixed quantity, then Eq. (5a) leads directly to the first integral

$$\sigma = \sigma_o \exp(K_\sigma\phi) \qquad (7)$$

wherein $K_\sigma \equiv -\frac{3}{4}\sin 2\theta$, and σ_o is the initial value of σ.

Making use of this result, then Eq. (5b) yields, as its "solution"

$$\tau_\sigma = [\tfrac{3}{2}(1+\cos 2\theta)-K_\sigma{}^2]\sigma \equiv K_\tau\sigma \qquad (8)$$

Next, recalling that $\phi \equiv \dot{\phi}t$, Eq. (7) can be manipulated to produce the time equation

$$t = \frac{1}{K_\sigma\dot{\phi}}\ln\frac{\sigma}{\sigma_o} = \frac{1}{K_\sigma\dot{\phi}}\ln\left(\frac{l}{l_o}\right) \qquad (9)$$

Finally, rewriting Eq. (5a) in terms of σ, i.e.,

$$\sigma' = -\tfrac{3}{4}\sigma\sin 2\theta \equiv K_\theta\sigma \qquad (10)$$

then Eqs. (7–10) describe *this* "extensible tethered bodies" problem, completely. Interestingly, one sees that the rate of "reeling-in" or "reeling-out" (σ') of the tether is linearly related to the length (σ); correspondingly, the required tensile load (for the tether) is also linearly related to the line length. However, contrary to this, the "time" required to reel-in or reel-out the line is logarithmic in the length.

Discussion

As an interesting observation, Eq. (10) indicates that the reeling-in of the tethered mass (m_2) can be accommodated *only* in the *first* and *third* θ-quadrants. Conversely, the line may be *reeled-out* in the second and fourth quadrants, only.

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